(ii) That the positive electricity within the atom is not in an electronic condition, but is distributed fairly uniformly through the atom.

III. Experiments are described on the absorption of the homogeneous β-rays. It is shown that the first stage in the absorption of a pencil of homogeneous β-rays consists in the scattering of the rays. The absorption of a completely scattered beam of homogeneous β-rays is shown to take place according to an exponential law.

In conclusion, I have much pleasure in once more recording my best thanks to Prof. Sir J. J. Thomson for much inspiration and advice during the course of these experiments.

Aerial Plane Waves of Finite Amplitude.

By Lord Rayleigh, O.M., F.R.S.

(Received July 8, 1910.)

Waves of Finite Amplitude without Dissipation.

In the investigations which follow, we are concerned with the motion of an elastic fluid in one dimension, say, parallel to x. It is implied not only that there are no component velocities perpendicular to x, but that the motion is the same in any perpendicular plane, so that it is a function of x and of the time (t) only. If u be the velocity at any point x, p the pressure, ρ the density, X an impressed force, the dynamical equation for an inviscid fluid is

$$\frac{du}{dt}+u\frac{du}{dx} = \mathrm{X}-\frac{1}{\rho}\frac{dp}{dx}. \tag{1}$$

At the same time the "equation of continuity" takes the form

$$\frac{d\rho}{dt}+\frac{d(\rho u)}{dx} = 0. \tag{2}$$

The first step, and it was a very important one, in the treatment of waves of finite amplitude is due to Poisson.* Under the assumption of Boyle's law, $p = a^2\rho$, he proved that for waves travelling in one direction (positive) the circumstances of the propagation are expressed by

$$u = f\{x-(a+u)t\}, \tag{3}$$

* "Mémoire sur la Théorie du Son," 'Journ. de l'École Polytechnique, 1808, vol. 7 p. 319.

in which f denotes an arbitrary function. When u can be neglected in comparison with a, this reduces to the familiar law of undisturbed propagation applicable to infinitesimal waves.

Poisson does not discuss the significance of (3) further than to show that the boundaries of a continuous wave, limited to a finite range along x, are propagated with the ordinary velocity a, and that accordingly the length of the wave does not alter as it advances. The meaning of (3) is that in general u advances with a velocity equal, not to a, but to $a+u$, and that this might be expected is very easily seen (Earnshaw). From the ordinary theory we know that an infinitely small disturbance is propagated with a certain velocity a, which velocity is relative to the parts of the medium undisturbed by the wave. Let us consider now the case of a wave so long that the variations of velocity and density are insensible for a considerable distance along it, and at a place where the velocity (u) is finite let us imagine a small secondary wave to be superposed. The velocity with which the secondary wave is propagated through the surrounding medium is a, but on account of the local motion of the medium itself the whole velocity of advance is $a+u$, and depends upon the part of the long wave at which the small wave is placed. What has been said of the secondary wave applies also to the parts of the long wave itself, and thus we see that after a time t the place where a certain velocity u is to be found is in advance of its original position by a distance equal, not to at, but to $(a+u)t$, or, as we may express it, u is propagated with velocity $(a+u)$.

A closer discussion of the solution represented by Poisson's integral was given by Stokes,* who pointed out the difficulty which ultimately arises from the motion becoming discontinuous. If we draw a curve to represent the distribution of velocity, taking x for abscissa and u for ordinate, we may find the corresponding curve after the lapse of time t by the following construction:—Through any point on the original curve draw a straight line in the positive direction parallel to x, and of length equal to $(a+u)t$, or, as we are concerned with the shape of the curve only, equal to ut. The locus of the ends of these lines is the velocity-curve after a time t.

But this law of derivation cannot hold good indefinitely. The crests of the velocity-curve gain continually on the troughs and must at last overtake them. After this the curve would indicate two values of u for one value of x, ceasing to represent anything that could actually take place. In fact we are not at liberty to push the application of the integral beyond the point at which the velocity becomes discontinuous, or the velocity-curve has a vertical tangent. In order to find when this happens, let us take two

* "On a Difficulty in the Theory of Sound," 'Phil. Mag.,' November, 1848.

neighbouring points on any part of the curve which slopes downwards in the positive direction, and inquire after what time this part of the curve becomes vertical. If the difference of abscissæ be dx, the hinder point will overtake the forward point in the time $-dx/du$. Thus the motion, as determined by Poisson's integral, becomes discontinuous after a time equal to the reciprocal, taken positively, of the greatest negative value of du/dx.

For example, let us suppose that

$$u = \mathrm{U}\cos\frac{2\pi}{\lambda}\{x-(a+u)\,t\}, \tag{4}$$

where U is the greatest initial velocity. When $t = 0$, the greatest negative value of du/dx is $-2\pi\mathrm{U}/\lambda$, so that discontinuity will commence at the time $t = \lambda/2\pi\mathrm{U}$.

The only kind of wave travelling in the positive direction which can escape ultimate discontinuity is one which has no forward slope. This is the case of a wave forming the transition between a larger constant value of u when x exceeds a certain value, and a smaller constant value when x falls short of a certain value. As time passes, the slope everywhere becomes easier. We shall see presently that this wave is a wave of rarefaction, in the sense that during its passage the gas passes from a greater to a less density.

It is worthy of remark that, although we may of course conceive a wave of finite disturbance to exist at any moment, there is in general a limit to the duration of its previous independent existence. By drawing lines in the negative instead of in the positive direction we may trace the history of the velocity-curve; and we see that as we push our inquiry further and further into past time the forward slopes become easier and the backward slopes steeper. At a time equal to the greatest positive value of dx/du, antecedent to that at which the curve is first contemplated, the velocity would be discontinuous. The exception is now a wave of condensation, involving a passage always from a less to a greater density.

When discontinuity sets in, a state of things exists to which the usual differential equations are inapplicable; and the subsequent progress of the motion has not been determined. It is probable, as suggested by Stokes, that some sort of reflection would ensue. In regard to this matter we must be careful to keep purely mathematical questions distinct from physical ones. We shall see later how the tendency to discontinuity may be held in check by forces of a dissipative character. But this has nothing directly to do with the mathematical problem of determining what would happen to waves of finite amplitude in a medium, free from viscosity, whose pressure is under all circumstances proportional to the density.* To suppose that the problem has

* 'Theory of Sound,' 1878, § 251.

no solution would seem to be tantamount to admitting an inherent contradiction in the assumption, usually made in hydrodynamics, of a continuous fluid subject to Boyle's law. It would be strange if the necessity of a molecular constitution for gases could be established by such an argument.

With Poisson's integral (3), showing how the velocity is propagated, there is associated another law connecting the velocity and the density in a positive progressive wave. In the case of a fluid obeying Boyle's law this relation is

$$u - a \log \rho = \text{const.} \tag{5}$$

It does not occur explicitly in Poisson's memoir, and Earnshaw considers that Poisson did not discover it. Certainly it is remarkable that he omitted to formulate the law, but at the same time it is difficult to suppose him ignorant of it, seeing that it follows by simple subtraction from two of his equations.* A formula equivalent to (5) was given explicitly, so far as I know, for the first time by Airy,† who attributes it to De Morgan.

The assumption that in a progressive wave there is a definite relation between u and ρ forms the basis of Earnshaw's investigation.‡ That such a relation is to be expected may be shown by a line of argument analogous to that already employed in connection with Poisson's integral.

Whatever may be the law of pressure as a function of density, the velocity of propagation of small disturbances is according to the usual theory equal to $\sqrt{(dp/d\rho)}$, and in a positive progressive wave the relation between velocity and condensation (s) is

$$u : s = \sqrt{(dp/d\rho)}, \tag{6}$$

where $s = \delta\rho/\rho$. If this relation be violated at any point, a wave will emerge, travelling in the negative direction. Let us now picture to ourselves the case of a positive progressive wave in which the changes of velocity and density are very gradual but become important by accumulation, and let us inquire what condition must be satisfied in order to prevent the formation of a negative wave. It is clear that the answer to the question whether or not a negative wave will be generated at any point will depend upon the state of things in the immediate neighbourhood of the point, and not upon the state of things at a distance from it, and will therefore be determined by the criterion applicable to small disturbances. In applying this criterion we are to consider the velocities and condensations, not absolutely, but relatively to

* Equations (1), p. 364, and (b), p. 367.

† 'Phil. Mag.,' 1849, vol. 34, p. 402. The corresponding formula for long tidal waves of finite amplitude was also given.

‡ 'Roy. Soc. Proc.,' January 6, 1859; 'Phil. Trans.,' 1860, p. 133.

those prevailing in the neighbouring parts of the medium, so that the form of (6) proper for the present purpose is

$$du = \sqrt{\left(\frac{dp}{d\rho}\right)} \cdot \frac{d\rho}{\rho}, \tag{7}$$

whence

$$u = \int \sqrt{\left(\frac{dp}{d\rho}\right)} \cdot \frac{d\rho}{\rho}, \tag{8}$$

which is the relation between u and ρ generally necessary for a positive progressive wave, as laid down by Earnshaw.*

Earnshaw worked with the so-called Lagrangian form of the equations, in which the motions of particular particles are followed, and he obtained complete solutions for a wave progressive in one direction. In the case of Boyle's law the relation between velocity and density is that already given (5), and in the case of the adiabatic law, where

$$\frac{p}{p_0} = \left(\frac{\rho}{\rho_0}\right)^\gamma, \tag{9}$$

Earnshaw finds

$$\left(\frac{\rho}{\rho_0}\right)^{\frac{1}{2}(\gamma-1)} = 1 + \frac{(\gamma-1)\,u}{2a}, \tag{10}$$

where a is the velocity of infinitesimal disturbances under the condition represented by p_0, ρ_0, viz. $a^2 = \gamma p_0/\rho_0$. In (9) γ denotes as usual the ratio of the two specific heats; and in (10), applicable to a *positive* progressive wave, the constant of integration has been so chosen that $u = 0$ corresponds to $\rho = \rho_0$.

The generalised form of Poisson's integral, appropriate when p is *any* given function of ρ, does not appear quite explicitly in Earnshaw's memoir. The line of argument already used shows that it must be

$$u = f[x - \{u + \sqrt{(dp/d\rho)}\}\, t]. \tag{11}$$

In the case of a gas obeying Boyle's law,

$$\sqrt{(dp/d\rho)} = \text{const.}, \tag{12}$$

and (11) reduces to Poisson's form.

In the case of the adiabatic law, we have from (9), (10),

$$\sqrt{\left(\frac{dp}{d\rho}\right)} = a\left(\frac{\rho}{\rho_0}\right)^{\frac{1}{2}(\gamma-1)} = a + \frac{\gamma-1}{2}u, \tag{13}$$

so that (Earnshaw)

$$u + \sqrt{(dp/d\rho)} = a + \tfrac{1}{2}(\gamma+1)\,u. \tag{14}$$

Thus (11) assumes the form

$$u = f[x - \{a + \tfrac{1}{2}(\gamma+1)\,u\}\, t], \tag{15}$$

and this with (10) may be considered to constitute the solution of the

* 'Theory of Sound,' 1878, § 251.

problem up to the point where discontinuity sets in. We may fall back upon Boyle's law by putting in (15) $\gamma = 1$.

It appears that whether the relation of pressure to density be isothermal or adiabatic, there is a change of type as the wave advances. There can be no escape from such a change unless $u + \sqrt{(dp/d\rho)}$ be constant. Using (8), we may deduce in this case

$$\sqrt{(dp/d\rho)} = \mathrm{B}/\rho,$$

B being a constant, whence

$$p = \mathrm{A} - \mathrm{B}^2/\rho \tag{16}$$

expresses the only law of pressure under which waves of finite amplitude can be propagated without undergoing a change of type (Earnshaw). A simpler derivation of (16) will be given presently.

Earnshaw further considers the genesis of disturbance in a gas originally at rest by the motion of a piston (supposed to be contained in a tube), but some of his conclusions appear to need revision. All that is required in these problems is virtually contained in (8), (11). If X denote the position of the piston at time T, the velocity of its motion is $\mathrm{U} = d\mathrm{X}/d\mathrm{T}$, and this velocity is shared by the gas in contact with it. On the positive side the velocity of propagation of U (equal to u) is, by (11),

$$\mathrm{U} + \sqrt{(dp/d\rho)};$$

so that if at time t (greater than T), U is to be found at x, we must have

$$x = \mathrm{X} + \{\mathrm{U} + \sqrt{(dp/d\rho)}\}(t - \mathrm{T}). \tag{17}$$

Among the problems which naturally suggest themselves would be to determine what happens when the piston originally at rest at $\mathrm{X} = 0$ begins to move at time $\mathrm{T} = 0$ with a constant velocity. But if this velocity be positive, the discontinuity, which immediately ensues, causes the failure of our equations. On the other hand, if the constant velocity be negative, say $-\mathrm{V}$, the initial discontinuity disappears forthwith, and the subsequent motion may be traced.

Take, for example, the case of Boyle's law. We have

$$\mathrm{X} = \mathrm{UT}, \qquad \mathrm{U} + \sqrt{(dp/d\rho)} = \mathrm{U} + a;$$

so that

$$x = \mathrm{U}t + a(t - \mathrm{T}) \qquad (t > \mathrm{T}),$$

and we have to consider where the velocities 0 and $-\mathrm{V}$ are to be found at time t. Now $\mathrm{U} = 0$ corresponds to the range of T from $-\infty$ to 0, so that x ranges from at to ∞. Again, $\mathrm{U} = -\mathrm{V}$ corresponds to the range of T from 0 to t, so that x ranges from $(a - \mathrm{V})t$ to $-\mathrm{V}t$. The whole range of x on the positive side of the piston is now accounted for, except the interval from $x = (a - \mathrm{V})t$ to $x = at$. This is occupied by the transition of velocity

from 0 to $-\mathrm{V}$, and we infer, from what was said in the discussion of Poisson's integral, that this transition must take place linearly.

Under Boyle's law the relation between velocity and density (5) is such that, however fast the piston may recede (u negative), a complete vacuum can never be formed behind it. It is otherwise under the adiabatic law (10), where $\rho = 0$ corresponds to

$$u = -2a/(\gamma-1) \quad \text{(Earnshaw).}$$

It may be of interest to consider further a few examples of (17). Still assuming Boyle's law, let us suppose that the piston is at rest ($\mathrm{X} = 0$) until $\mathrm{T} = 0$, and then moves with uniform acceleration (g), so that ($\mathrm{T}+$)

$$\mathrm{U} = g\mathrm{T}, \qquad \mathrm{X} = \tfrac{1}{2} g\mathrm{T}^2 = \mathrm{U}^2/2g.$$

The use of these in (17) gives

$$x = \frac{a^2}{2g} + (\mathrm{U}+a)\, t - \frac{(\mathrm{U}+a)^2}{2g}, \tag{18}$$

showing that the relation between x and U is parabolic. In (18) we see that $dx/d\mathrm{U}$ vanishes, when $t = (\mathrm{U}+a)/g$. Thus, if g be positive, *i.e.* if the wave be one of condensation, discontinuity sets in at the front after an interval, reckoned from the beginning of the motion, equal to a/g. But if g be negative, there is no discontinuity, and (18) remains valid for an indefinite time.

In general, as in the last example ($g+$), the discontinuity sets in locally at one point of the velocity-curve, while other parts are temporarily exempt. It is of interest to inquire under what law the piston must advance so as to generate a linear velocity-curve. For then, under the adiabatic law, which includes Boyle's, a velocity-curve, once linear, remains linear, and if discontinuity enters, it must affect the whole curve simultaneously.

It may be worth while to pause here for a moment to inquire what law of pressure is implied in the permanence of the linear character of the velocity-curve. By (7)

$$\rho \frac{du}{d\rho} = \sqrt{\left(\frac{dp}{d\rho}\right)};$$

so that if $u+\sqrt{(dp/d\rho)}$ is a linear function of u, $du/d\log\rho$ must also be a linear function of u. This requires that

$$\rho\, du/d\rho = \mathrm{C}\rho^n,$$

where C and n are constants, and the most general relation between p and ρ consistent with the requirements is

$$p = \mathrm{A} + \mathrm{B}\rho^\gamma, \tag{19}$$

where A, B, γ, are constants. The relation (19) may be regarded as a kind

of generalised adiabatic law; it includes the special law (16) under which a velocity-curve is absolutely permanent in type.

Supposing the motion to commence at $T = 0$, we have

$$X = \int_0^T U\,dT = UT - \int_0^U T\,dU,$$

and hence from (17), under the supposition of Boyle's law,

$$x = -\int T\,dU - (U+a)\,t - aT\,; \qquad (20)$$

and the question before us is so to determine T as a function of U that (20) may be linear in U. From (20) when t is constant

$$\frac{dx}{dU} = -T + t - a\frac{dT}{dU} = t_0, \text{ say},$$

where t_0 is a constant, whence

$$T = t - t_0 + He^{-U/a},$$

H being the constant of integration. But, since $U = 0$ when $T = 0$, this assumes the form

$$T = T'(1 - e^{-U/a}), \qquad (21)$$

T' being written for $t - t_0$. Or, if we express U in terms of T,

$$U = -a \log\left(1 - \frac{T}{T'}\right). \qquad (22)$$

In (21), (22), T' is positive, if U is positive; and U becomes infinite when $T = T'$. We must therefore regard the law as limited to values of T less than T'.

From (22) we find

$$X = \int_0^T U\,dT = a\left\{(T'-T)\log\left(1 - \frac{T}{T'}\right) + T\right\}, \qquad (23)$$

which completely expresses the motion of the piston. The corresponding velocity-curve at time t may be verified by means of (20), (21). It is expressed by

$$U = \frac{at - x}{T' - t}, \qquad (24)$$

exhibiting the linear character of the slope of velocity. Evidently the slope becomes vertical throughout when $t = T'$.

In the above example it is not necessary to suppose the law of motion of the piston continued up to $T = T'$. On the contrary, we may imagine U to increase up to some prescribed finite value and then to remain constant. In this case the slope expressed by (24) forms the transition between $U = 0$ for values of x greater than at and the finite value at which the acceleration of the piston stops.

If the wave be one of rarefaction we must take T′ negative, say $-\mathrm{T}''$. In this case the analogue of (22) shows that U constantly increases in numerical value but does not become infinite in any finite time. The analogue of (24) is

$$\mathrm{U} = \frac{x-at}{t+\mathrm{T}''}, \tag{25}$$

representing a slope which ever grows easier as time passes.

The problem also admits of solution when the gas follows the adiabatic law (9). As in (15), (20),

$$x = \mathrm{X} + \{a + \tfrac{1}{2}(\gamma+1)\,\mathrm{U}\}\,(t-\mathrm{T})\,; \tag{26}$$

and (26) is to satisfy the condition of making $dx/d\mathrm{U}$ constant. In this

$$d\mathrm{X}/d\mathrm{U} = \mathrm{U}\,.\,d\mathrm{T}/d\mathrm{U},$$

so that
$$\frac{d\mathrm{T}}{d\mathrm{U}}\{(\gamma-1)\,\mathrm{U}+2a\}+(\gamma+1)(\mathrm{T}-t) = \text{const.}$$

On integration we obtain, under the condition that U and T vanish together,

$$\left(1-\frac{\mathrm{T}}{\mathrm{T}'}\right)\left[1+\frac{\gamma-1}{2a}\,\mathrm{U}\right]^{\frac{\gamma+1}{\gamma-1}} = 1, \tag{27}$$

T′ being a constant which, if positive, corresponds to $\mathrm{U} = \infty$, thereby determining U as a function of T.

For the value of $\mathrm{X} = \int_0^{\mathrm{T}} \mathrm{U}\,d\mathrm{T}$, we have

$$\begin{aligned}\frac{\gamma-1}{2a}\frac{\mathrm{X}}{\mathrm{T}'} &= -\frac{\gamma+1}{2}\left[\left(1-\frac{\mathrm{T}}{\mathrm{T}'}\right)^{\frac{2}{\gamma+1}}-1\right]-\frac{\mathrm{T}}{\mathrm{T}'}\\ &= \frac{\gamma-1}{2}-\frac{\gamma+1}{2}\left(1+\frac{\gamma-1}{2a}\,\mathrm{U}\right)^{-\frac{2}{\gamma-1}}+\left(1+\frac{\gamma-1}{2a}\,\mathrm{U}\right)^{-\frac{\gamma+1}{\gamma-1}}.\end{aligned} \tag{28}$$

It may be observed that, although U is infinite when $\mathrm{T} = \mathrm{T}'$, X remains finite. Using this in (26), we get finally, on reduction,

$$x = at - \tfrac{1}{2}(\gamma+1)\,\mathrm{U}\,(\mathrm{T}'-t), \tag{29}$$

or
$$\mathrm{U} = \frac{2}{\gamma+1}\,\frac{at-x}{\mathrm{T}'-t}. \tag{30}$$

If $x < at$, U is a linear function of x, and (T′ being positive) the slope of the velocity curve increases until it becomes vertical when $t = \mathrm{T}'$.

If T′ is negative, the wave is one of rarefaction, and (30) applies however great t may be.

By putting $\gamma = 1$ we fall back from (30) to (24), and less simply from (28) to (23).*

* I have since found that this problem was successfully treated by Hugoniot.

Riemann's work* is of somewhat later date than Earnshaw's, but in one important respect is more general. It may be convenient briefly to recall the principal result.

Taking p a given function of ρ, say $\phi(\rho)$, and putting $X = 0$, we have from (1) and (2)

$$\frac{du}{dt}+u\frac{du}{dx} = -\phi'(\rho)\frac{d\log\rho}{dx}, \qquad \frac{d\log\rho}{dt}+u\frac{d\log\rho}{dx} = -\frac{du}{dx}.$$

If the second of these equations be multiplied by $\pm\sqrt{\{\phi'(\rho)\}}$ and be added to the first, we find

$$\frac{dr}{dt} = -\{u+\sqrt{\phi'(\rho)}\}\frac{dr}{dx}, \qquad \frac{ds}{dt} = -\{u-\sqrt{\phi'(\rho)}\}\frac{ds}{dx}, \tag{31}$$

where
$$2r = f(\rho)+u, \qquad 2s = f(\rho)-u, \tag{32}$$

and
$$f(\rho) = \int\sqrt{\phi'(\rho)}\,.\,d\log\rho. \tag{33}$$

From these follow

$$dr = \frac{dr}{dx}[dx-\{u+\sqrt{\phi'(\rho)}\}\,dt]$$

$$ds = \frac{ds}{dx}[dx-\{u-\sqrt{\phi'(\rho)}\}\,dt];$$

so that r remains constant when x and t change in such a manner that $dx = \{u+\sqrt{\phi'(\rho)}\}dt$, and s remains constant when x and t change so that $dx = \{u-\sqrt{\phi'(\rho)}\}dt$. In the case of a positive progressive wave $s = 0$, whence $f(\rho) = u$ and also $r = u$. The velocity with which u travels in such a wave is accordingly $u+\sqrt{(dp/d\rho)}$, of which fact (11) is merely another form of statement. Riemann's equations are more general than anything previously given, as not limited to a single progressive wave.

Since Riemann's equations do not seem to have been applied in any example of continuous motion, I have thought it worth while to inquire whether they can be satisfied when r and s are both linear functions of x, Boyle's law being assumed, so that in (32), (33)

$$\sqrt{\phi'(\rho)} = a, \qquad f(\rho) = a\log\rho.$$

If we suppose

$$r = Ax+B, \qquad s = Cx+D, \tag{34}$$

we obtain, on substitution in (31), equations for the determination of A, B, C, D, as functions of the time. In the first instance we find

$$A-C = 1/t, \tag{35}$$

in which to t a constant may be added, and further

$$A = \frac{H+1}{2t}, \qquad C = \frac{H-1}{2t}, \tag{36}$$

* 'Göttingen Abhandlungen,' 1860, vol. 8.

when H is an arbitrary constant. Also

$$B - D = -aH + L/t, \tag{37}$$

L being another arbitrary constant, and thence

$$B = \tfrac{1}{2}a(H^2 - 1)\log t + \frac{L(H+1)}{2t} + \tfrac{1}{2}M, \tag{38}$$

$$D = \tfrac{1}{2}a(H^2 - 1)\log t + \frac{L(H-1)}{2t} + \tfrac{1}{2}N, \tag{39}$$

with

$$M - N = -2aH. \tag{40}$$

If we allow the origin of x to be arbitrary, as well as that of t, we may write

$$2r = (H+1)x/t + (H^2 - 1)a\log t + M, \tag{41}$$

$$2s = (H-1)x/t + (H^2 - 1)a\log t + N, \tag{42}$$

$$u = r - s = x/t - aH, \tag{43}$$

$$a\log\rho = r + s = Hx/t + (H^2 - 1)a\log t + \tfrac{1}{2}(M+N). \tag{44}$$

If $H = \pm 1$, the logarithmic term disappears, and either r or s is constant. In these cases we fall back upon single progressive waves.

If $H = 0$, $u = x/t$ in (43), and (44) gives $1/\rho$ proportional to t. The density is thus uniform with respect to x, but the volume of a given mass grows proportionally with t. The uniform expansion occurs in such a manner that the gas remains unmoved at the origin of co-ordinates. Since in this case $du/dt + u\,du/dx = 0$, we see that every part of the gas moves with unaccelerated velocity.

Waves of Permanent Regime.

When waves are propagated in one dimension without change of type, the circumstances are dynamically the same as in steady motion, as appears at once by impressing on the system a velocity equal and opposite to that of wave-propagation. The problem may conveniently be considered under this form.

From the general equation of continuity (2), by making $d\rho/dt$ equal to zero, or independently, we have

$$\rho u = \rho_0 u_0 = m, \tag{45}$$

where m is a constant which Rankine called the mass-velocity. The dynamical equation (1) reduces to

$$u\frac{du}{dx} + \frac{1}{\rho}\frac{dp}{dx} = X. \tag{46}$$

If X = 0, (46) may be written

$$\int_{p_0}^{p} \frac{dp}{\rho} = \tfrac{1}{2}u_0^2 - \tfrac{1}{2}u^2, \tag{47}$$

where u_0 is the velocity corresponding to p_0.

Eliminating u, we get

$$\int_{p_0}^{p} \frac{dp}{\rho} = \tfrac{1}{2}u_0^2\left(1-\frac{\rho_0^2}{\rho^2}\right), \tag{48}$$

determining the law of pressure under which alone it is possible for a stationary wave to maintain itself in fluid moving (outside the wave) with velocity u_0. From (48)

$$\frac{dp}{d\rho} = u_0^2\frac{\rho_0^2}{\rho^2}, \tag{49}$$

or

$$p + m^2/\rho = p_0 + m^2/\rho_0, \tag{50}$$

the law found by Earnshaw.

Since, under the adiabatic law, the relation between density and pressure differs from (50), we conclude that a self-maintaining stationary aerial wave is an impossibility, unless it be in virtue of impressed forces, or of viscosity, or other dissipative agencies not now regarded.

When the changes of density concerned are *small*, (50) may be satisfied approximately; and we see from (49) that the velocity of the stream (outside the wave) necessary to keep the wave stationary is given by

$$u_0 = \sqrt{(dp/d\rho)},$$

which is the same as the velocity of the wave reckoned relatively to the fluid at a distance.

This way of regarding the subject shows, perhaps more clearly than any other, the nature of the relation between velocity and density. In a stationary wave-form a loss of velocity accompanies an augmented density, according to the principle of energy, and therefore the fluid composing the condensed parts of a wave moves forward more slowly than the undisturbed portions. Relatively to the fluid at a distance, the motion of the condensed parts is in the same direction as that in which the waves travel.

By means of (46), we can find what impressed force is required in order to ensure a stationary wave-form when (50) is not satisfied. For example, if $p = a^2\rho$, we find from (45), (46),

$$X = u\frac{du}{dx} + a^2\frac{d\log\rho}{dx} = (u^2-a^2)\frac{d\log u}{dx}, \tag{51}$$

showing that an impressed force is necessary at every place where u is variable and unequal to a. In (51) X is the accelerating force so called.

The actual force operative upon the element of mass $\rho\, dx$ is $X\rho\, dx$. Thus, on integration,

$$\int X\rho\, dx = m\int\left(1-\frac{a^2}{u^2}\right)du = m\,(u_2-u_1)\left\{1-\frac{a^2}{u_1u_2}\right\}, \tag{52}$$

if the range of integration extend from the place where the velocity is u_1 to the place where it becomes equal to u_2. The integral applied force vanishes if the terminal velocities are such that their geometric mean is a. We may apply this to the case of a velocity-curve giving a simple gradual transition from one constant velocity u_1 to another constant velocity u_2. Under the above condition the integral force vanishes, but finite forces are required at all points of the slope, except the particular point where $u = a$.

It is of some importance to notice that although, under the condition $u_1u_2 = a^2$, the applied forces contribute on the whole no *momentum*, yet they do contribute *energy*, positive or negative. To find the work done in unit of time by the forces we have

$$2/m\,.\int X\rho\,.\,u\,.\,dx = u_2{}^2-u_1{}^2-2u_1u_2\log(u_2/u_1). \tag{53}$$

The better to interpret this let us suppose that u_1 and u_2 are positive, and in the first instance that $u_2>u_1$, so that the fluid passes from a less to a greater velocity, or by (45) from a greater to a less density. In the case of a wave of rarefaction we have therefore to consider the sign of

$$y^2-1-2y\log y, \tag{54}$$

when $y>1$. It is not difficult to prove that this sign is always positive. When $y-1$ is small, the approximate value of (54) is $\frac{1}{3}(y-1)^3$, and is therefore positive when $y>1$. Again, if we remove the positive factor y from (54) and then differentiate, we obtain $(1-1/y)^2$, which is positive. Hence, when $y>1$, (54) is necessarily positive. The propagation of the wave of rarefaction without change of type requires that the impressed forces, contributing on the whole no momentum, should nevertheless do work upon, *i.e.* communicate energy to, the gas.

In like manner, if the wave be one of condensation, *i.e.*, if the gas passes from a less to a greater density, the operation of the impressed forces is to remove energy from the gas forming the wave. It follows that although dissipative forces, such as those arising from viscosity, may possibly constitute a machinery capable of maintaining the type of a wave of condensation, in no case can they maintain the type of a wave of rarefaction.

It is desirable to extend this argument to waves propagated under the adiabatic law. In general, from (45), (46),

$$X\rho = m\frac{du}{dx}+\frac{dp}{dx};$$

so that $\quad\int X\rho\, dx = m\,(u_2-u_1)+p_2-p_1 = m^2/\rho_2-m^2/\rho_1+p_2-p_1.$

As in (50), the condition that on the whole no momentum is communicated is

$$m^2/\rho_2 - m^2/\rho_1 + p_2 - p_1 = 0. \tag{55}$$

Again,

$$m^{-1}.\int X\rho.u.dx = \tfrac{1}{2}(u_2^2 - u_1^2) + \int_{p_1}^{p_2} \frac{dp}{\rho} = \int_{p_1}^{p_2} \frac{dp}{\rho} - \frac{p_2 - p_1}{2}\left(\frac{1}{\rho_2} + \frac{1}{\rho_1}\right). \tag{56}$$

The question in which we are interested is the sign of (56). If we regard v (the volume of unit mass), viz., $1/\rho$, as the ordinate, and p as the abscissa of a curve, the first term on the right of (56) represents the area of the curve bounded by two ordinates and the axis of p, while the second term is what the area would be if the ordinate retained throughout the mean of the terminal values.

So far the argument is general. If the relation between p and v be adiabatic, and $p_2 > p_1$, the expression (56) is negative, since v proportional to $p^{-1/\gamma}$ makes d^2v/dp^2 positive.* The final pressure exceeding the initial pressure denotes a wave of condensation, and we conclude, as before, that maintenance of type in such a wave requires removal of energy from the wave, while in the contrary case of a wave of rarefaction additional energy would need to be supplied.

The problem now under discussion is closely related to one which has given rise to a serious difference of opinion. In his paper of 1848 already referred to, Stokes considered the *sudden* transition from one constant velocity to another, and concluded that the necessary conditions for a permanent regime could be satisfied. Results equivalent to his may be deduced from (45) in connection with the condition ($u_1 u_2 = a^2$) already found from (52) to express that there is no change of momentum on the whole. Thus,

$$u_1 = a\sqrt{(\rho_2/\rho_1)}, \qquad u_2 = a\sqrt{(\rho_1/\rho_2)}. \tag{57}$$

Similar conclusions were put forward by Riemann in 1860 (*loc. cit.*). Commenting on these results in the "Theory of Sound" (1878), I pointed out that although the conditions of *mass* and *momentum* were satisfied, the condition of *energy* was violated, and that therefore the motion was not possible; and in republishing this paper† Stokes admitted the criticism, which had indeed already been made privately by Kelvin. On the other hand, Burton‡ and H. Weber§ maintain, at least to some extent, the original view.

* Compare Lamb's 'Hydrodynamics,' § 280.

† 'Collected Works,' vol. 2, p. 55.

‡ 'Phil. Mag.,' 1893, vol. 35, p. 316.

§ 'Die Partiellen Differentialgleichungen der Mathematischen Physik,' Braunschweig, 1901, vol. 2, p. 496.

Inasmuch as they ignored the question of energy, it was natural that Stokes and Riemann made no distinction between the cases where energy is gained or lost. As I understand, Weber abandons Riemann's solution for the discontinuous wave (or *bore,* as it is sometimes called for brevity) of rarefaction, but still maintains it for the case of the bore of condensation. No doubt there is an important distinction between the two cases; nevertheless, I fail to understand how a loss of energy can be admitted in a motion which is supposed to be subject to the isothermal or adiabatic laws, in which no dissipative action is contemplated. In the present paper the discussion proceeds upon the supposition of a *gradual* transition between the two velocities or densities. It does not appear how a solution which violates mechanical principles, however rapid the transition, can become valid when the transition is supposed to become absolutely abrupt. All that I am able to admit is that under these circumstances dissipative forces (such as viscosity) that are infinitely small may be competent to produce a finite effect.

If we suppose that under the influence of small dissipative forces the bore of Stokes and Riemann can be propagated, at least approximately, we naturally inquire whether it can be regarded as the complete outcome of the simple progressive wave with a straight velocity slope which, as we have found, tends after a definite interval of time to assume the character of a bore. It would seem that the answer must be in the negative. Taking Boyle's law, we recognise from (5) that in the progressive wave, just before the formation of the bore, the relation between the velocities and densities is

$$u_2 - u_1 = a \log \rho_2 - a \log \rho_1,$$

while in (57) the relation is

$$u_2 - u_1 = a\sqrt{(\rho_2/\rho_1)} - a\sqrt{(\rho_1/\rho_2)}.$$

The two functions of ρ on the right, which are independent of any common addition to u_1 and u_2, cannot be identified (unless $\rho_2 = \rho_1$), as we have found already in discussing (54). This incompatibility may be regarded as a confirmation of Stokes' opinion that something of the nature of reflection must ensue.

Permanent Regime under the influence of Dissipative Forces.

The first investigation to be considered under this head is a very remarkable one by Rankine "On the Thermodynamic Theory of Waves of Finite Longitudinal Disturbance,"* which (except a limited part expounded

* 'Phil. Trans.,' 1870, vol. 160, Part II, p. 277.

by Maxwell in his "Theory of Heat") has been much neglected.* Conduction of heat is here for the first time taken into account and although there are one or two serious deficiencies, not to say errors, presently to be noticed, the memoir marks a very definite advance.

The first step is the establishment of an equation equivalent (when the wave is reduced to rest) to (45), and of Earnshaw's relation (50), in which equations we shall usually substitute v, the volume of unit mass, for $1/\rho$. Rankine remarks that "no substance yet known fulfils the condition expressed by (50) between finite limits of disturbance, at a constant temperature, nor in a state of non-conduction of heat (called the *adiabatic* state). In order, then, that permanency of type may be possible in a wave of longitudinal disturbance, there must be both change of temperature and conduction of heat during the disturbance." However, we shall see later that even under Boyle's law *viscosity* is competent to endow a wave with permanency.

The question is, how can Earnshaw's law be satisfied? Obviously not (in the absence of viscosity) if the expansions are adiabatic; but if at every stage the right quantity of heat is added or subtracted, the gas may be made to follow any prescribed law. This is the idea underlying Rankine's investigation. For the unit mass of a *perfect gas* we have, as usual, $pv = \mathrm{R}\theta$, θ denoting absolute temperature. The condition of the gas is defined by any two of the three quantities p, v, θ, and the third may be expressed in terms of them. The relation between simultaneous variations is

$$d\theta/\theta = dp/p + dv/v. \tag{58}$$

In order to effect the changes specified by dp and dv, it is in general necessary to communicate heat to the gas. Calling the necessary quantity of heat $d\mathrm{Q}$, we may write

$$d\mathrm{Q} = \left(\frac{d\mathrm{Q}}{dv}\right) dv + \left(\frac{d\mathrm{Q}}{dp}\right) dp. \tag{59}$$

Suppose now (a) that $dp = 0$. Equations (58), (59), give

$$\frac{d\mathrm{Q}}{d\theta}\,(p \text{ constant}) = \left(\frac{d\mathrm{Q}}{dv}\right)\frac{v}{\theta},$$

where $d\mathrm{Q}/d\theta$ (p constant) expresses the specific heat of the gas under constant pressure. Denoting this by C, we have

$$\mathrm{C} = \left(\frac{d\mathrm{Q}}{dv}\right)\frac{v}{\theta}.$$

* I must take my share of the blame. Rankine is referred to by Lamb ('Hydrodynamics,' 1906, p. 466). The body of Rankine's memoir seems to have been composed without acquaintance with the writings of his predecessors; but in a supplement he notices the work of Poisson, Stokes, Airy, and Earnshaw.

Again, suppose (b) that $dv = 0$. We find in a similar manner that if c denote the specific heat under constant volume,

$$c = \left(\frac{dQ}{dp}\right)\frac{p}{\theta}.$$

Thus, in general, $$dQ/\theta = C\,dv/v + c\,dp/p. \tag{60}$$

If between (58) and (60) we eliminate dp, there results

$$dQ = (C-c)\frac{p\,dv}{R} + c\,d\theta. \tag{61}$$

In (61) $dQ = 0$ corresponds to adiabatic expansion, when according to Mayer's principle the cooling effect $-\,c\,d\theta$ is equal to the external work done by the gas $p\,dv$. Hence

$$C - c = R, \tag{62}$$

and therefore $$\gamma = \frac{C}{c} = \frac{C}{C-R}, \tag{63}$$

a relation discovered by Rankine himself in 1850.

Rankine then applies (60) to find what heat must be communicated in order that the gas may follow Earnshaw's law making $dp = -\,m^2 dv$. With regard to (62), it appears that

$$dQ = \frac{dp}{m^2(\gamma-1)}\{p_0 + m^2 v_0 - (\gamma+1)\,p\}. \tag{64}$$

It will be understood that under the condition now imposed of Earnshaw's relation, as well as of the ordinary gas law, there remains but one independent variable, and that the state of the gas may be expressed in terms of any *one* of the three quantities p, v, θ.

We have next to consider how far the necessary supply of heat defined by (64) can be effected by *conduction*. If the initial state (distinguished by suffix 1) and final state (with suffix 2) be of uniformity with respect to x, the total quantity of heat received by the gas during its passage must be zero, or $\int dQ = 0$. Hence from (64) Rankine finds

$$p_0 + m^2 v_0 = \tfrac{1}{2}(\gamma+1)(p_1 + p_2). \tag{65}$$

This is a necessary condition; but of course there is nothing so far to show that it is sufficient.

In (65) p_0, v_0 are any corresponding values of p and v within the wave, and we may identify them with p_1, v_1.*

Thus

$$m^2 v_1 = \tfrac{1}{2}(\gamma-1)\,p_1 + \tfrac{1}{2}(\gamma+1)\,p_2. \tag{66}$$

* We may, of course, also identify p_0, v_0 with p_2, v_2.

The velocity u_1, equal to mv_1, is that with which the wave advances relatively to the fluid in state (1). And

$$u_1^2 = m^2 v_1^2 = v_1 \{\tfrac{1}{2}(\gamma-1)p_1 + \tfrac{1}{2}(\gamma+1)p_2\} \tag{67}$$

gives the square of the velocity of wave-propagation relatively to fluid (1). The velocity of propagation of infinitely small disturbances (p_2 nearly equal to p_1) is given by $u_1^2 = \gamma p_1 v_1$; and thus a wave of finite condensation is propagated faster than an infinitesimal wave, and according to (67) a wave of finite rarefaction would be propagated slower than an infinitesimal wave. Moreover, there is no limit to the velocity of a wave of condensation.

Rankine proceeds to express the absolute temperature (θ) at a point where the pressure is p in a wave of permanent type. By Earnshaw's law (50) in combination with (65)

$$\frac{\theta}{\theta_0} = \frac{pv}{p_0 v_0} = \frac{p}{p_0} \cdot \frac{(\gamma+1)(p_1+p_2)-2p}{(\gamma+1)(p_1+p_2)-2p_0}, \tag{68}$$

and for the ratio of terminal temperatures

$$\frac{\theta_2}{\theta_1} = \frac{p_2}{p_1} \cdot \frac{(\gamma+1)p_1+(\gamma-1)p_2}{(\gamma+1)p_2+(\gamma-1)p_1}. \tag{69}$$

The second fraction on the right of (69) obviously represents the ratio of volumes v_2/v_1, or of densities ρ_1/ρ_2.

In order to justify (65), it is not necessary that the terminal states be states of absolute uniformity. It will suffice that the temperature be there stationary ($d\theta/dx = 0$), which secures that no conduction of heat takes place there, and a state of stationary temperature usually involves a stationary pressure. To make the most of (65) we must apply it to the smallest ranges, *i.e.* between consecutive places where dp/dx vanishes.

But here a question arises which Rankine does not seem to have considered. In order to secure the necessary transfers of heat by means of conduction it is an indispensable condition that the heat should pass from the hotter to the colder body. If maintenance of type be possible in a particular wave as the result of conduction, a reversal of the motion will give a wave whose type cannot be so maintained. We have seen reason already for the conclusion that a dissipative agency can serve to maintain the type only when the gas passes from a less to a more condensed state. If this be so, the application which Rankine makes to a periodic wave is evidently prohibited.

According to the *second* law of thermodynamics, the criterion whether the transformation is possible as the result of dissipative action is the sign of

$\int dQ/\theta$. If this be negative, the transformation is not possible. From (64) with use of (65)

$$dQ = \frac{(\gamma+1)\,dp}{2m^2(\gamma-1)}(p_1+p_2-2p). \tag{70}$$

In (68) we may give p_0, θ_0 any corresponding values found in the wave. Thus p_0 lies between p_1 and p_2, and $(\gamma>1)$

$$(\gamma+1)(p_1+p_2)-2p_0 \text{ is positive.}$$

Accordingly $\int dQ/\theta$ takes the same sign as

$$\int_{p_1}^{p_2} \frac{dp\,(p_1+p_2-p)}{p\,\{(\gamma+1)(p_1+p_2)-2p\}}. \tag{71}$$

The integral (71) is evaluated without difficulty. Dropping the factor $1/(\gamma+1)$, and writing $\varpi = p_2/p_1$, we get

$$\log\varpi+\gamma\log\frac{\gamma+1+(\gamma-1)\varpi}{\gamma-1+(\gamma+1)\varpi}. \tag{72}$$

It is evident that (72) changes sign when we substitute $1/\varpi$ for ϖ.

If we expand (72) in powers of $(\varpi-1)$ we find

$$(72) = \frac{(\gamma^2-1)(\varpi-1)^3}{12\gamma^2}+\ldots,$$

the terms in $(\varpi-1)$, $(\varpi-1)^2$, disappearing. Thus, when $\varpi-1$ is positive, (72) begins positive. Differentiating (72) with respect to ϖ, we get

$$\frac{1}{\varpi}+\frac{\gamma(\gamma-1)}{\gamma+1+(\gamma-1)\varpi}-\frac{\gamma(\gamma+1)}{\gamma-1+(\gamma+1)\varpi}. \tag{73}$$

When (73) is reduced to a single fraction, the denominator is positive, and the numerator is

$$(\gamma^2-1)(\varpi-1)^2.$$

We infer that if $\varpi>1$, (72) is always positive, and that if $\varpi<1$, (72) is always negative. Hence if $p_2>p_1$, *i.e.* if the wave be one of condensation, the communications of heat required are such as may arise from conduction; but if the wave be one of rarefaction, its permanency can in no wise be attained as the result of conduction. A wave of condensation here means a wave such that during its progress the gas passes always from a less dense to a more dense state, and the most important case is when the limits are finite, so that the passage constitutes the transition from one uniform density to a greater uniform density.

Rankine proceeds to examine more particularly under what conditions a wave can be permanent. "In order that a particular type of disturbance may be capable of permanence during its propagation, a relation must exist between the temperatures of the particles and their relative positions, such

that the conduction of heat between the particles may effect the transfers of heat required by the thermodynamic conditions of permanence of type."

The equation of conduction is readily found. The heat conducted in unit time across a layer of the gas is represented by $k\, d\theta/dx$, where k is a coefficient of conductivity which may be a function of the condition of the gas, here dependent on one variable. The equation of conduction is ($u = +$)

$$v\frac{d}{dx}\left(k\frac{d\theta}{dx}\right) = \frac{\mathrm{D}}{\mathrm{D}t}\,\mathrm{Q} = u\frac{d\mathrm{Q}}{dx} = mv\frac{d\mathrm{Q}}{dx},$$

whence, if we reckon Q from the initial condition of constant pressure p_1,

$$k\frac{d\theta}{dx} = m\mathrm{Q}. \tag{74}$$

And from (70)

$$\mathrm{Q} = \frac{\gamma+1}{2m^2(\gamma-1)}\int_{p_1}^{p}(p_1+p_2-2p)\,dp = \frac{\gamma+1}{2m^2(\gamma-1)}(p-p_1)(p_2-p). \tag{75}$$

Also from (50), (65),

$$p+m^2v = \tfrac{1}{2}(\gamma+1)(p_1+p_2), \tag{76}$$

whence

$$\theta = \frac{pv}{\mathrm{R}} = \frac{p}{2m^2\mathrm{R}}\{(\gamma+1)(p_1+p_2)-2p\}, \tag{77}$$

and

$$\frac{d\theta}{dp} = \frac{(\gamma+1)(p_1+p_2)-4p}{2m^2\mathrm{R}}. \tag{78}$$

Using these, we find with regard to (62)

$$dx = \frac{k}{m\mathrm{Q}}\frac{d\theta}{dp}\,dp = \frac{k\,dp}{mc(\gamma+1)}\,\frac{(\gamma+1)(p_1+p_2)-4p}{(p-p_1)(p_2-p)}, \tag{79}$$

by which is determined the distribution of pressure (and thence of density and temperature) along the line of propagation.

On the supposition that k is constant, Rankine integrates (79) in terms of logarithms. Writing

$$p-\tfrac{1}{2}(p_1+p_2) = q, \qquad \tfrac{1}{2}(p_2-p_1) = q_1,$$

he obtains

$$\frac{dx}{dq} = \frac{k}{mc(\gamma+1)}\,\frac{(\gamma-1)(p_1+p_2)-q}{q_1^2-q^2},$$

and

$$x = \frac{k}{mc(\gamma+1)}\left\{\frac{(\gamma-1)(p_1+p_2)}{2q_1}\log\frac{q_1+q}{q_1-q}+2\log\left(1-\frac{q^2}{q_1^2}\right)\right\}, \tag{80}$$

x being measured from the place where $q = 0$. Mathematically the wave is infinitely long; but practically the transition of pressure is effected in a distance comparable with $k/mc(\gamma+1)$, which may be small in terms of ordinary standards. It is to be observed that the general character of the result does not depend upon the constancy of k.

Reverting to (79), we see that the denominator on the right is positive, and

that the numerator is also positive for that part of the wave where p is nearly equal to $\frac{1}{2}(p_1+p_2)$. Thus, for this part of the wave at any rate, p and x increase together; or, since u is positive, the gas passes to a condition of greater density—the wave must be one of condensation. This consideration, as we have seen, Rankine overlooked. And a further limitation presents itself: since there cannot be two pressures in one place, it is evident that dx/dp must not change sign. The numerator in (79) must be positive over its *whole* range from p_1 to p_2, and this will not be the case if p_2/p_1 exceed $(\gamma+1)/(3-\gamma)$, equal for common gases to 1·61. The conclusion is that the only kind of wave, involving a transition from one uniform pressure to another, which can be maintained with the aid of conduction is a wave of condensation, and then only when the ratio of pressures does not exceed a moderate value.

The next contribution to the subject upon which I have to comment is contained in a long and ably written memoir by Hugoniot.* This author, though he covers to a great extent the same ground, makes no reference to Stokes, Earnshaw, Riemann, or Rankine, and but a very slight one to Poisson—a circumstance which increases the difficulty of comparison. Since Hugoniot uses the Lagrangian form of equation, his investigation runs naturally on the same lines as Earnshaw's, whose general solution for a single progressive wave is reproduced. I have already alluded to the solution of special problems relating to the propagation of a wave of variable type.

The most original part of Hugoniot's work has been supposed to be his treatment of discontinuous waves involving a sudden change of pressure, with respect to which he formulated a law often called after his name by French writers. But a little examination reveals that this law is *precisely the same* as that given 15 years earlier by Rankine, a fact which is the more surprising inasmuch as the two authors start from quite different points of view. Rankine's investigation, as we have seen, is expressly based upon conduction of heat in the gas, but Hugoniot supposes his gas to be non-conducting. A question of some delicacy is here involved, which will repay careful examination. It will be convenient to give a paraphrase of Hugoniot's argument.†

This argument depends upon an application of the principle of energy to a region bounded by two fixed planes, including the place of discontinuity. The work done by the fluid as it emerges with volume v_2 against the pressure p_2 is

* 'Journal de l'École Polytechnique,' 1887, 1889.
† Compare Lamb's 'Hydrodynamics,' 1906, § 280.

p_2v_2. On the whole, therefore, the external work done by the passage of the unit of mass is $p_2v_2-p_1v_1$. The increase of kinetic energy of the fluid is

$$\tfrac{1}{2}(u_2^2-u_1^2) = \tfrac{1}{2}m^2(v_2^2-v_1^2) = \tfrac{1}{2}(v_2+v_1)(p_1-p_2),$$

in virtue of (50), which requires that

$$p_1-p_2+m^2(v_1-v_2) = 0. \tag{81}$$

The sum of these is

$$p_2v_2-p_1v_1+\tfrac{1}{2}(p_1-p_2)(v_2+v_1) = \tfrac{1}{2}(v_2-v_1)(p_2+p_1). \tag{82}$$

We have next to consider the internal energy of unit of mass in the initial and final states. For this purpose we suppose the gas to expand adiabatically from its actual volume v to an infinite volume. In this expansion the work done by the gas is

$$\int_v^\infty p\,dv = \frac{pv}{\gamma-1}, \tag{83}$$

so that the difference of internal energy in the two states is

$$\frac{p_2v_2}{\gamma-1}-\frac{p_1v_1}{\gamma-1}. \tag{84}$$

The principle of energy requires that the sum of this and (82) be zero, whence

$$\gamma = \frac{(p_2-p_1)(v_1+v_2)}{(p_1+p_2)(v_1-v_2)} \tag{85}$$

is the relation between the pressures and volumes in the two states. The result thus found by Hugoniot is the same as Rankine's. From Rankine's equation (65)

$$p_1+m^2v_1 = p_2+m^2v_2 = \tfrac{1}{2}(\gamma+1)(p_1+p_2) = \tfrac{1}{2}(p_1+p_2)+\tfrac{1}{2}m^2(v_1+v_2),$$

it follows that
$$m^2 = \gamma\frac{p_1+p_2}{v_1+v_2} = \frac{p_2-p_1}{v_1-v_2}, \tag{86}$$
which is identical with (85).

The first remark that I will make is that, although Hugoniot assumes that the transition between the two states is sudden, there is nothing in his argument which requires this, all that is really necessary being that the *régime* is permanent. The next remark is that, however valid (85) may be, its fulfilment does not secure that the wave so defined is possible. As a matter of fact, a whole class of such waves is certainly impossible, and I would maintain, further, that a wave of the kind is never possible under the conditions, laid down by Hugoniot, of no viscosity or heat-conduction.

A closer examination of the process by which (85) was obtained will show that while the first law of thermodynamics has been observed, the second law has been disregarded. The crux of the matter lies in the comparison

of the internal energies of the incoming and outgoing gas expressed in (84). If (p_2, v_2) and (p_1, v_1) lie upon the same adiabatic, the work corresponding to the passage from the one state to the other is given without ambiguity by (84). But in the present case the two states do not lie upon the same adiabatic, and the work required is deduced upon the assumption that nothing is involved in the passage at $v = \infty$ from one adiabatic to the other. What is actually there required is the communication (positive or negative) of an infinitesimal quantity of heat. From the point of view of the first law the infinitesimal quantity of heat may be neglected, but not so from the point of view of the second law, since the transfer is supposed to take place at the zero of temperature. When heat and work are distinguished, infinitesimal heat at zero may have a finite value. The imaginary passage to infinity has the advantage of leading rapidly to the required conclusion, but it rather tends to obscure the real nature of the process. While all the other items of the account are mechanical work, the passage from one adiabatic to the other (which may take place at constant finite volume) is a question of *heat* as distinguished from work. If during a complete cycle work would be lost and corresponding heat gained, the operation is dissipative and there need be no contradiction if viscosity or heat-conduction enter, but the opposite contingency of a gain of work at the expense of heat is excluded in all cases. The conclusion is the same as before. While a wave of condensation may, perhaps, maintain a permanent regime as the result of dissipative agencies, a permanent wave of rarefaction is excluded.

It is remarked by Hugoniot that even when the ratio p_1/p_2 is infinite, v_2/v_1 does not exceed $(\gamma+1)/(\gamma-1)$, which for common gases is equal to about 6. A similar remark is made by Duhem,* who discusses the whole question with great generality. With regard to perfect gases "lorsqu'une quasi-onde de choc se propage au sein d'un gas parfait, le fluide le plus condensé est toujours en amont de l'onde et le fluide le moins condensé en aval." But, so far as I see, neither of these authors proves that the propagation is possible in any case.

It is a question of great interest to inquire what is the influence of viscosity and especially whether alone, or in co-operation with heat-conduction, it allows a wave of condensation to acquire a permanent regime. We proceed to consider this question on the basis of the usual equations, although it must be admitted that their application to conditions which are somewhat extreme raises points of uncertainty.

Reverting to our original equations, we recognise that (45) is unaffected

* 'Zeitschrift f. Physikal. Chem.,' 1909, vol. 69, p. 169.

by the inclusion of viscosity, and that the change required in (46) is represented by writing*

$$X = \frac{4}{3\rho}\frac{d}{dx}\left(\mu\frac{du}{dx}\right),$$

so that (46) takes the form

$$m\frac{du}{dx}+\frac{dp}{dx}-\frac{4}{3}\frac{d}{dx}\left(\mu\frac{du}{dx}\right) = 0,$$

whence ($v = 1/\rho$)

$$p+m^2v-\frac{4}{3}m\mu\frac{dv}{dx} = p_1+m^2v_1 = p_2+m^2v_2, \tag{87}$$

the terminal states (p_1, v_1), (p_2, v_2), being of uniformity, so that dv/dx there vanishes. From this it appears that (81), relating to the terminal states, holds good equally when viscosity is regarded.

A simple example under the head of viscosity is to suppose the temperature maintained uniform, as by a powerful radiation, so that the gas follows Boyle's law, making

$$pv = p_1v_1 = p_2v_2 = a^2.$$

From this and (87) we get

$$m^2v_1v_2 = a^2,$$

and

$$p_1+m^2v_1 = a^2/v_1+a^2/v_2.$$

Using these in (87), we find

$$\frac{3m\,dx}{4\mu} = -\frac{v\,dv}{(v_1-v)(v-v_2)}, \tag{88}$$

as governing the distribution of v along the line of propagation. In a wave of condensation $v_1 > v > v_2$, so that the denominator on the right of (88) is positive. Thus when m is positive, dv/dx is negative, as should be the case. On integration (μ constant)

$$\frac{3mx}{4\mu} = \frac{1}{v_1-v_2}\{v_1\log(v_1-v)-v_2\log(v-v_2)\}, \tag{89}$$

the origin of x being chosen suitably.

The transition of volumes from v_1 to v_2 occupies, mathematically speaking, the whole range from $x = -\infty$ to $x = +\infty$, but practically it may be very sudden. Since in (88) dx/dv never changes sign, the condition of permanency for a condensational wave can always be satisfied, whatever may be the value of the ratio v_1/v_2† or p_1/p_2, contrasting in this respect with the limitation found to be necessary on Rankine's conclusion relative to heat-conduction.

* Lamb's 'Hydrodynamics,' §§ 314, 316.

† But the limitation pointed out by Hugoniot still obtains; otherwise, one of the pressures would be negative.

As regards the velocity of wave propagation into the rarer medium, we have for its square

$$u_1^2 = m^2 v_1^2 = a^2 v_1/v_2. \tag{90}$$

Returning to the case where heat development and viscosity are both regarded, we see that in virtue of (81) Hugoniot's reasoning is still applicable without change, and it leads to the same final relation (85) as was found by Rankine when heat-conduction is alone considered.

In endeavouring to apply Rankine's method to the more general case where viscosity is retained, we shall find it more convenient to treat v, or $(1/\rho)$, rather than p, as independent variable. If, as before, dQ denotes the total quantity of heat received by unit mass of the gas, we have from (60), (62),

$$(\gamma-1)\frac{dQ}{dx} = \gamma p\frac{dv}{dx}+v\frac{dp}{dx};$$

or, on elimination of p by means of (87),

$$(\gamma-1)\frac{dQ}{dx} = \gamma\left\{p_1+m^2v_1-m^2v+\tfrac{4}{3}m\mu\frac{dv}{dx}\right\}\frac{dv}{dx} +v\left\{-m^2\frac{dv}{dx}+\tfrac{4}{3}m\frac{d}{dx}\left(\mu\frac{dv}{dx}\right)\right\}. \tag{91}$$

In (91) dQ consists of two parts, the first (dQ_1), with which alone Rankine dealt, the heat received by conduction, and the second (dQ_2) the heat developed internally under viscosity. As regards the latter, the heat developed in volume v and time dt is $\frac{4}{3}v\mu\,(du/dx)^2dt$,* in which we are to replace dt by dx/u, and u by mv, so that

$$\frac{dQ_2}{dx} = \tfrac{4}{3}m\mu\left(\frac{dv}{dx}\right)^2. \tag{92}$$

Multiplying this by $(\gamma-1)$ and subtracting it from (91), we get

$$(\gamma-1)\frac{dQ_1}{dx} = \gamma\,(p_1+m^2v_1)\frac{dv}{dx}-(\gamma+1)\,m^2v\frac{dv}{dx}+\tfrac{4}{3}m\frac{d}{dx}\left(\mu v\frac{dv}{dx}\right). \tag{93}$$

As in Rankine's investigation, the whole heat received by conduction in passing from one uniform state v_1 to another uniform state v_2 must vanish. Hence, on integrating between these limits, and dividing out the factor (v_2-v_1), we have

$$(\gamma+1)\,m^2(v_1+v_2) = 2\gamma\,(p_1+m^2v_1) = 2\gamma\,(p_2+m^2v_2)$$
$$= \gamma\,(p_1+p_2)+\gamma m^2(v_1+v_2),$$

or, as in (86), $$m^2(v_1+v_2) = \gamma\,(p_1+p_2),$$

the same relation as was found by Rankine. Introducing it into (93), we get

$$(\gamma-1)\frac{dQ_1}{dx} = \tfrac{1}{2}(\gamma+1)\,m^2\left\{(v_1+v_2)\frac{dv}{dx}-\frac{dv^2}{dx}\right\}+\tfrac{4}{3}m\frac{d}{dx}\left(\mu v\frac{dv}{dx}\right). \tag{94}$$

* Lamb's 'Hydrodynamics,' § 341.

A particular case arises when we suppose the conductivity to be zero, so that dQ_1/dx vanishes throughout. We have then

$$\tfrac{4}{3}\mu v \frac{dv}{dx} + \tfrac{1}{2}(\gamma+1)\, m\,(v_1 - v)\,(v - v_2) = 0, \tag{95}$$

differing from (88) only by the factor $\frac{1}{2}(\gamma+1)$. On the supposition that μ is constant, the solution is nearly the same as in (89), and, in fact, reduces to it when $\gamma = 1$, which represents Boyle's law. This case of no conduction is thus satisfactorily disposed of. Whatever be the ratio of pressures, a wave of condensation is always possible.

It should be remarked, however, that the supposition of constant μ does consist with the facts as known for actual gases when γ differs from unity. For such gases viscosity, though independent of *density*, varies with *temperature*, so that μ will not be constant in (95). But since μ is always positive, this complication merely affects the particular form of the integral and not the general conclusion as to the possibility of a permanent wave.

In general, from (94), if we reckon Q_1 from the terminal state v_1,

$$(\gamma-1)\, Q_1 = \tfrac{1}{2}(\gamma+1)\, m^2\,(v_1 - v)\,(v - v_2) + \tfrac{4}{3} m\mu v \frac{dv}{dx}. \tag{96}$$

The equation of conduction is the same (74) as before. And for θ, from (87),

$$\theta = \frac{pv}{R} = \frac{v}{R}\left\{\frac{\gamma+1}{2\gamma}\, m^2\,(v_1+v_2) - m^2 v + \tfrac{4}{3} m\mu \frac{dv}{dx}\right\};$$

so that with regard to (62) the equation of conduction becomes

$$\frac{k}{mc}\left[m\frac{dv}{dx}\left\{\frac{\gamma+1}{2\gamma}(v_1+v_2) - 2v\right\} + \tfrac{4}{3}\frac{d}{dx}\left(\mu v \frac{dv}{dx}\right)\right]$$
$$= \tfrac{1}{2}(\gamma+1)\, m\,(v_1 - v)\,(v - v_2) + \tfrac{4}{3}\mu v \frac{dv}{dx}. \tag{97}$$

By omitting the terms containing μ we may of course fall back on Rankine's problem.

Equation (97), in its general form, is much more complicated than when either viscosity or heat-conduction is alone regarded, in consequence of the occurrence of the differential coefficient of the second order. In general, both k and μ are functions of temperature, and therefore of v; but, according to Maxwell's theory, which assumes a molecular repulsion inversely as the fifth power of the distance, $c\mu/k$ is independent of temperature (as well as of density), and takes the value $\frac{2}{5}$. And it would seem that this independence of temperature and density is general, seeing that the ratio is of no dimensions, at least so long as the repulsive force can be represented by an inverse power of the distance.* We shall write h for the above ratio and

* Compare 'Roy. Soc. Proc.,' 1900, vol. 66, p. 68; 'Scientific Papers,' vol. 4, p. 453.

assume that for a given gas it is an absolute constant. Thus, μ' being written for μ/m, (97) takes the form—

$$\mu' \frac{d}{dx}\left(\mu' \frac{dv^2}{dx}\right)+\mu' \frac{dv^2}{dx}\left\{\frac{3(\gamma+1)}{8\gamma}\frac{v_1+v_2}{v}-\tfrac{3}{2}-h\right\} = \tfrac{3}{4}h(\gamma+1)(v_1-v)(v-v_2); \tag{98}$$

in which v^2 may be regarded as the dependent variable. For v^2 we shall write ξ, and if

$$\mathrm{U} = \mu' \, d\xi/dx, \tag{99}$$

our equation, since it contains x only through dx, may be reduced to one of the first order in U and ξ, *i.e.*,

$$\mathrm{U}\frac{d\mathrm{U}}{d\xi}+\mathrm{U}f(\xi) = \mathrm{F}(\xi), \tag{100}$$

where
$$f(\xi) = \frac{3(\gamma+1)}{8\gamma}\frac{\sqrt{\xi_1}+\sqrt{\xi_2}}{\sqrt{\xi}}-\tfrac{3}{2}-h, \tag{101}$$

and
$$\mathrm{F}(\xi) = \tfrac{3}{4}h(\gamma+1)(\sqrt{\xi_1}-\sqrt{\xi})(\sqrt{\xi}-\sqrt{\xi_2}). \tag{102}$$

If U can be found as a function of ξ from (100), x follows by simple integration of (99).

In considering equation (100) we may conveniently regard ξ as the linear co-ordinate of a material particle of unit mass moving in a straight line with velocity U. The first term, $\mathrm{U}d\mathrm{U}/d\xi$, then represents the acceleration of the particle; the second, $\mathrm{U}f(\xi)$, may be regarded as a *resistance*, proportional to the velocity, and at the same time variable with the position (ξ); and the third term on the right hand represents a force, which is also a function of position. If t be the time in this subsidiary problem, $\mathrm{U} = d\xi/dt$, and (100) may be written

$$\frac{d^2\xi}{dt^2}+f(\xi)\frac{d\xi}{dt} = \mathrm{F}(\xi), \tag{103}$$

while by (99)
$$dx = \mu' \, dt. \tag{104}$$

If μ' be constant, the substitution of $\mu' t$ for x in (98) is obvious.

If we take $h = 0{\cdot}4$, $\gamma = 1{\cdot}41$, (101), (102), become

$$f(\xi) = 0{\cdot}641\frac{\sqrt{\xi_1}+\sqrt{\xi_2}}{\sqrt{\xi}}-1{\cdot}900, \tag{105}$$

$$\mathrm{F}(\xi) = 0{\cdot}723(\sqrt{\xi_1}-\sqrt{\xi})(\sqrt{\xi}-\sqrt{\xi_2}). \tag{106}$$

It will be observed that, over the range from ξ_1 to ξ_2, $\mathrm{F}(\xi)$ is positive, but that the sign of $f(\xi)$ is doubtful. If ξ has the greater terminal value ξ_1, f is negative; but, when it has the smaller terminal value ξ_2, the sign depends upon the ratio ξ_1/ξ_2. If this ratio $<1{\cdot}21$, f is negative; otherwise it is positive.

I suppose that a complete analytical solution of our equation is not to be expected, and it is, indeed, hardly necessary for our purpose. What we most wish to know is whether a solution is possible which satisfies the prescribed conditions. Among these is the requirement that U in (100) vanish at both limits; and even then the manner of evanescence must be such as to secure that x, as determined by (99), shall be infinite at these limits. As the problem originally presents itself, we should have the representative particle travelling in the negative direction from ξ_1 to ξ_2, starting with no velocity and arriving with no velocity. It seems simpler to consider it in a modified form, *i.e.* with the motion reversed, so that it takes place in the direction of the force F. There is, then, no question of the particle stopping between the limits and returning upon its course. We may make this change, if in (105) we reverse the sign of f. We consider, then, the motion of the particle to be in the positive direction, from ξ_2 to ξ_1, with zero velocity at both limits, the motion between ξ_2 and ξ_1 being aided by the force F, which itself vanishes at these limits, and being also subject to a force of the nature of resistance, proportional to velocity. When ξ_1/ξ_2 does not exceed 1·21, the force is a resistance in the ordinary sense, *i.e.* it opposes the motion, and, in any case, it has this character near (and beyond) the arrival end ξ_1. But when ξ_1/ξ_2 exceeds 1·21, the force becomes what we may call a counter-resistance, and aids the motion near the initial end ξ_2. As regards F, in the neighbourhood of each limit it becomes a force proportional to distance therefrom, repulsive near ξ_1, and attractive near ξ_2. Thus, when ξ is nearly equal to ξ_2,

$$F(\xi) = 0{\cdot}723\,\frac{\sqrt{\xi_1}-\sqrt{\xi_1}}{2\sqrt{\xi_2}}(\xi-\xi_2); \tag{107}$$

and when ξ is nearly equal to ξ_1,

$$F(\xi) = 0{\cdot}723\,\frac{\sqrt{\xi_1}-\sqrt{\xi_2}}{2\sqrt{\xi_1}}(\xi_1-\xi). \tag{108}$$

The particle, starting from ξ_2, is bound to go through to ξ_1. If it arrives at ξ_1 with zero velocity, we shall have, presumably, a solution of our problem. It is possible, however, that on first arrival at ξ_1, it may pass through, and only settle down after a number of oscillations. To this there does not appear to be any objection; but if on the return from ξ_1 it overshoots ξ_2, it can never again return to ξ_1, since on the left of ξ_2 the sign of f is negative; and then our problem has no solution. On the other hand, from the nature of F, it is not possible for the particle passing through ξ_1 in the positive direction to escape returning.

The character of the start from ξ_2 can be investigated with the aid of approximate equations. Thus, in (100) we may treat $f(\xi)$ as constant, say

2α, where α may be either positive or negative, and take, as in (107), $F(\xi) = \beta(\xi - \xi_2)$, where β is positive, so that

$$U\frac{dU}{d\xi} + 2\alpha U - \beta(\xi - \xi_2) = 0. \tag{109}$$

If in (109) we assume

$$U = \lambda(\xi - \xi_2), \tag{110}$$

we find that the equation is satisfied provided that

$$\lambda = -\alpha \pm \sqrt{(\alpha^2 + \beta)}, \tag{111}$$

one value (λ_1) being positive and one (λ_2) negative. In the present case, where U must be positive when $\xi > \xi_2$, λ_1 is to be chosen.

The differential equation (109) can be made homogeneous, and its general solution,* when λ is real, can be put into the form

$$\frac{\{U - \lambda_1(\xi - \xi_2)\}^{\lambda_1}}{\{U - \lambda_2(\xi - \xi_2)\}^{\lambda_2}} = C, \tag{112}$$

when C is an arbitrary constant. This solution, of course, covers the cases where the particle starts from ξ_2 with a finite velocity (U_0), and it appears that $C = U_0^{\lambda_1 - \lambda_2}$. We might conclude from this that when $U_0 = 0$, then $C = 0$, but the conclusion is not safe. If, however, U and $\xi - \xi_2$ are of the same order of magnitude, we may write $U = r(\xi - \xi_2)$, where r is not infinite. Substituting this in (112), we get

$$(r - \lambda_1)^{\lambda_1} . (r - \lambda_2)^{-\lambda_2} . (\xi - \xi_2)^{\lambda_1 - \lambda_2} = C. \tag{113}$$

When $\xi - \xi_2$ vanishes, the third factor on the left is zero, and, since the first and second factors are not infinite, the conclusion follows that $C = 0$. This takes us back to (110), (111), the second solution (involving λ_2) relating to the case where U is negative, the motion being one of *approach* from the positive side to ξ_2.

These conclusions may be arrived at more easily from (103), of which the general solution in the present case is

$$\xi - \xi_2 = Ae^{\lambda_1 t} + Be^{\lambda_2 t}, \tag{114}$$

giving

$$U = \lambda_1 Ae^{\lambda_1 t} + \lambda_2 Be^{\lambda_2 t}.$$

From these we may deduce (115)

$$U - \lambda_1(\xi - \xi_2) = (\lambda_2 - \lambda_1)Be^{\lambda_2 t},$$
$$U - \lambda_2(\xi - \xi_2) = (\lambda_1 - \lambda_2)Ae^{\lambda_1 t};$$

whence (112) follows by elimination of t. For our present purpose, $\xi - \xi_2$ is to vanish when $t = -\infty$, so that $B = 0$; and (110) follows with $\lambda = \lambda_1$. It will be remarked that (110) makes x, as determined by (99), infinite when

* See, for example, Boole's 'Differential Equations,' p. 33.

$\xi = \xi_2$. The circumstances of the start from ξ_2 are thus definite and suitable. The question is as to the arrival at ξ_1.

In the neighbourhood of ξ_1 the approximate equation is

$$U\frac{dU}{d\xi}+2\alpha' U-\beta'(\xi-\xi_1) = 0, \tag{116}$$

where α' is positive and, by (108), β' negative. If now

$$U = \lambda'(\xi-\xi_1), \tag{117}$$

the values of λ' are $$\lambda' = -\alpha' \pm \surd(\alpha'^2+\beta'); \tag{118}$$

so that both values, if real, are negative. On the supposition of reality, (112) retains its form (with ξ_1 for ξ_2). If the velocity of arrival (U_0) be finite, $C = U_0^{\lambda'_1-\lambda'_2}$, as before; and it might be supposed that, if U_0 vanishes when $\xi-\xi_1 = 0$, C would have to vanish or become infinite. Such a conclusion would be incorrect. If in (113) we suppose λ'_1 to be numerically the smaller of the two values, the third factor indeed vanishes with $(\xi-\xi_1)$ as before; but the conclusion that $C = 0$ is evaded if ultimately $r = \lambda'_1$.

The situation is most easily understood from the solution in terms of t as in (114), (115). Since λ'_1, λ'_2 are *both* negative, the condition that $\xi-\xi_1$ and U shall vanish together when $t = \infty$ is satisfied, whatever may be the values of A and B. There are now an infinite number of possible types of solution, instead of only one as in the former case. And it appears that the two simple types included under (117) are not at all upon an equal footing. Except in the *particular* case where $A = 0$, the solution always tends ultimately to the form $U = \lambda_1(\xi-\xi_1)$, and of course it may assume this form throughout. All these solutions satisfy the condition as to the infinitude of x when $U = 0$.

Whether the values of λ' be real or not, the particle must ultimately settle down at ξ_1, unless it escape from the region to which the approximate equation applies. For in (118) the *real part* of λ' is always negative.

Returning to (100) in its general form, let us consider the variation of U for a given ξ as dependent upon variations in f and F. We have

$$U\frac{d\delta U}{d\xi}+\delta U\frac{dU}{d\xi}+f.\delta U+U.\delta f-\delta F = 0,$$

or $$\frac{d\delta U}{d\xi}+P\delta U-Q = 0, \tag{119}$$

where P and Q are supposed to be known functions of ξ, viz.,

$$P = \frac{f+dU/d\xi}{U}, \qquad Q = \frac{\delta F-U.\delta f}{U}. \tag{120}$$

The solution of the linear equation (119) is

$$\delta U = e^{-\int P d\xi}\left(\int e^{\int P d\xi} Q\, d\xi+c\right). \tag{121}$$

If $Q = 0$, δU has the same sign as c, so that an increment of velocity communicated at any point remains throughout of the same sign. Again, if Q be throughout of one sign, δU, as dependent upon it, has the same sign. For example, if U be positive over the range considered, δf positive, and δF negative, then δU is certainly negative. The increments δf, δF may be local, vanishing over any part of the range.

The application to the present problem is obvious. If the particle passing any point between ξ_2 and ξ_1, with velocity U, arrives at ξ_1 for the first time without velocity, it will still arrive at ξ_1 without velocity (it must in any case arrive, since F is positive), if U be diminished, or if f be increased, or if F be diminished, or if all these changes occur together. And in the limit, when ξ_1 is closely approached, the ratio of U to $(\xi_1 - \xi)$ is in general the same.

By use of this principle we may assure ourselves as to the possibility of a solution in certain cases where ξ_1/ξ_2 does not greatly exceed unity. We imagine a simplified problem which admits of analytical solution and is derived from the actual one by alterations which everywhere (over the range from ξ_2 to ξ_1) increase F and diminish f. If this modified problem admits of the solution required, *a fortiori* will the original problem do so.

If we consider the curve which according to (106) represents F as a function of ξ, we see that it is concave downwards, and that F will everywhere be increased if we substitute for the curve the two terminal tangents at ξ_2 and ξ_1, whose equations are given in (107), (108). The abscissa of K, the point of intersection, is $\xi = \sqrt{(\xi_2 \xi_1)}$. (Fig. 1.)

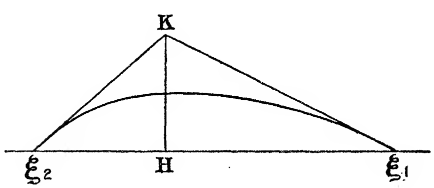

FIG. 1.

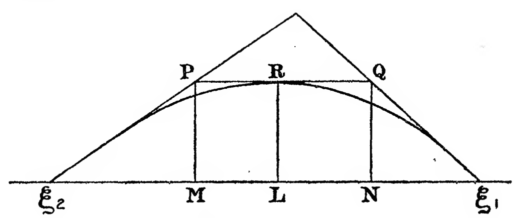

FIG. 2.

As regards f, its value is given by (105) with sign reversed, and is diminished when ξ is diminished. The changes will therefore be in the required direction if we represent f over ξ_2H by its value at ξ_2, viz.,

$$f_2 = 2\alpha = 1\cdot259 - 0\cdot641 s, \tag{122}$$

and from H to ξ_1 by its value at H, viz.,

$$f_1 = 2\alpha' = 1\cdot900 - 0\cdot641 (s^{\frac{1}{2}} + s^{-\frac{1}{2}}), \tag{123}$$

if for brevity we write s for $\sqrt{(\xi_1/\xi_2)}$, so that s is the ratio of terminal densities in the original problem.

As regards the first portion of the course, the solution already given (110), (111), determines the value of U on arrival at H. We have

$$U = \lambda_1 \xi_2 (s-1), \quad \text{where} \quad \lambda_1 = \sqrt{(\alpha^2+\beta)}-\alpha,$$

in which α is given by (122) while by (107) $\beta = 0{\cdot}361(s-1)$. Using these, we get

$$U = (s-1)\xi_2[\sqrt{\{(0{\cdot}630-0{\cdot}320s)^2+0{\cdot}361(s-1)\}}-0{\cdot}630+0{\cdot}320s]. \quad (124)$$

If $s = 1+\sigma$, where σ is small, (124) becomes

$$U = \sigma\xi_2 \times \frac{\beta}{2\alpha} = 0{\cdot}584\,\xi_2\sigma^2. \quad (125)$$

As regards the second portion of the course, the appropriate solution is provided by (117), (118), where λ' is restricted to be real. And in accordance with our suppositions α' is given by (123) while from (108) $\beta' = -0{\cdot}361(1-s^{-1})$. In choosing between the values of λ' we are at liberty to take that which gives the largest value of U at H consistent with $U = 0$ at ξ_1, viz.,

$$\lambda'_2 = -\alpha'-\sqrt{(\alpha'^2+\beta')}.$$

Thus at H $$U = (s^2-s)\,\xi_2\,\{\alpha'+\sqrt{(\alpha'^2+\beta')}\}, \quad (126)$$

or approximately, in terms of σ (supposed small),

$$U = \sigma\xi_2(0{\cdot}618+0{\cdot}033\sigma). \quad (127)$$

From (125), (127) we may infer that when σ is small, *i.e.* when s does not much exceed unity, the particle, starting from ξ_2, arrives at H with a velocity small enough to admit of its being stopped on arrival at ξ_1, even under the simplifying conditions that have been imposed, and therefore *a fortiori* under the actual conditions of the problem. Hence when ξ_1 does not too much exceed ξ_2, a wave of permanent regime *is* possible as the result of viscosity and heat-conduction.

But the range of ξ_1/ξ_2 thus proved admissible is rather severely limited. The postulated reality of λ' requires that $\alpha'^2+\beta'$ be positive, and this again requires that $s<1{\cdot}34$. For values of s greater than this the motion in the simplified problem would become oscillatory. Calculation shows that, for $s = 1{\cdot}34$, (124) gives

$$U = (s-1)\,\xi_2 \times 0{\cdot}20,$$

while (126) gives $$U = (s-1)\,\xi_2 \times 0{\cdot}43\,;$$

so that up to this limit the particle starting from ξ_2 in the simplified problem, and therefore also in the actual problem, would arrive at ξ_1 with zero velocity. Up to a ratio of densities equal to 1·34, the wave of permanent regime is certainly possible.

The next step in this line of procedure will be to replace the curve

representing F by the broken line $\xi_2 PQ \xi_1$ formed by *three* of its tangents, of which two are the same terminal tangents as before, while the third, PQ, may be taken to be the horizontal tangent parallel to $\xi_1\xi_2$. (Fig. 2.) By (106) the point L where F is a maximum is determined by

$$\sqrt{\xi} = \tfrac{1}{2}(\sqrt{\xi_1}+\sqrt{\xi_2}) = \tfrac{1}{2}\sqrt{\xi_2}(s+1),$$

and the corresponding value of F is

$$0{\cdot}1875\,\xi_2(s-1)^2.$$

The abscissæ of the points of intersection of the horizontal tangent with the terminal tangents are given by (107), (108). For M

$$\xi = \tfrac{1}{2}\xi_2(s+1),$$

and for N

$$\xi = \tfrac{1}{2}\xi_2(s^2+s).$$

As for f we are to take along $\xi_2 M$ the value at ξ_2; along MN the value at M; and along $N\xi_1$ the value at N, as given by (105), with sign changed.

For the two terminal portions the solutions are of the same form as before, but for the middle portion a new form is required. Making F constant in (100) and writing $2\alpha''$ for f, we find on integration

$$-4\alpha''^2\xi = F\log(F-2\alpha''U)+2\alpha''U+C, \tag{128}$$

where C is an arbitrary constant, to be determined so as to suit the velocity with which the particle arrives at M.

As before, the limiting value of s is determined by the consideration that, for our purpose, the arrival at ξ_1 must not be oscillatory. We get $s = 1{\cdot}633$ about, and with this value of s it is not difficult to show that the velocity of arrival at N is below the value prescribed by the solution for $N\xi_1$. The necessary conditions are thus fulfilled and we infer that the wave of uniform regime is possible so long as $s<1{\cdot}63$. Although this ratio is moderate, it exceeds that found admissible in Rankine's problem, which leaves viscosity out of account. We there found that the greatest admissible ratio of *pressures* was 1·61, which by (68) corresponds to 1·40 for the ratio of *densities*. Of course, in Rankine's problem the solution definitely fails at this point, while in the present problem all that we have so far proved is that the limit exceeds 1·63.

From the low limiting values of s found necessary in these two cases in order to secure the reality of the roots of (118), it might be inferred that the reality would fail for the limiting motion in the neighbourhood of ξ_1, when s had a considerable value, but such is not the case. If we use the value of f from (105) appropriate to the terminal point ξ_1 itself, we have

$$2\alpha' = 1{\cdot}900-0{\cdot}641\,(1+s^{-1}) = 1{\cdot}259-0{\cdot}641s^{-1};$$

and

$$\beta' = -0{\cdot}3615\,(1-s^{-1}).$$

Thus, even when $s = \infty$,

$$\alpha'^2+\beta' = 0{\cdot}3963-0{\cdot}3615 = +0{\cdot}0048;$$

and the roots are always real. Hence, subsidence at ξ_1 is not ultimately oscillatory, but there is nothing in this argument to exclude a finite number of oscillations before subsidence into the region governed by the approximate equation.

The method of approximation already followed might be pushed further, but it seems preferable to use the general method of numerical calculation for the solution of differential equations as formulated by Runge.* The equation is that numbered (100), in which f (whose sign is to be reversed) and F are given by (105), (106). If we write $U' = U/\xi_2$, $\xi' = \xi/\xi_2$, our equation takes the form

$$\frac{dU'}{d\xi'} = \frac{0{\cdot}723}{U'}(s-\sqrt{\xi'})(\sqrt{\xi'}-1)-1{\cdot}900+\frac{0{\cdot}641\,(1+s)}{\sqrt{\xi'}}. \tag{129}$$

The value of s being given, it is required to trace the connection between U' and ξ', simultaneous values being denoted respectively by a and b. If a receive the increment h, we have to calculate the corresponding increment k for b. If we call the function on the right of (129) $\phi\,(\xi', U')$, Runge gives as a first approximation to k

$$k_1 = \phi\,(a+\tfrac{1}{2}h, b+\tfrac{1}{2}\phi_0.h)\,h, \tag{130}$$

where $\phi_0 = \phi\,(a, b)$. The next approximation is

$$k = k_1+\tfrac{1}{3}(k_2-k_1), \tag{131}$$

where $$k_2 = \tfrac{1}{2}(k'+k''') \quad \text{and} \quad k' = \phi_0.h,$$

$$k'' = \phi\,(a+h, b+k')\,h, \qquad k''' = \phi\,(a+h, b+k'')\,h. \tag{132}$$

Having determined the new simultaneous values $a+h$, $b+k$, we make a fresh departure therefrom, and so trace out the function step by step.

In the present application the starting point is $a = 1$ (*i.e.* $\xi = \xi_2$), $b = 0$, and we have to trace the function until $\xi' = s^2$. The initial value of ϕ is to be found from (129) by putting $\xi' = 1$. Writing $U' = \phi_0\,(\xi'-1)$, we get

$$\phi_0^2+\{1{\cdot}900-0{\cdot}641\,(1+s)\}\,\phi_0-\tfrac{1}{2}\times 0{\cdot}723\,(s-1) = 0, \tag{133}$$

of which the positive root is to be chosen.

The extreme admissible value of s in our present problem is 6, and the first and rather elaborate calculation that I have made relates to this case. From (133) $\phi_0 = 3{\cdot}1591$, so that taking $a = 1$, $b = 0$, $h = 1$, we have $k' = 3{\cdot}1591$. Calculating from (130) we find $k_1 = 2{\cdot}255$, and from (132)

$$k'' = 1{\cdot}7076, \qquad k''' = 2{\cdot}0772,$$

* See Forsyth's 'Differential Equations,' p. 51.

making $k_2 = 2{\cdot}6182$. Hence the correction to k_1, viz., $\frac{1}{3}(k_2-k_1)$, is equal to $0{\cdot}121$, and $k = 2{\cdot}376$. Thus, corresponding to $\xi' = 2$, we get $U' = 2{\cdot}376$. The following are the values of U' obtained successively in this way:—

ξ'.	U'.	ξ'.	U'.	ξ'.	U'.
1	0·000	14	7·502	33	1·2942
2	2·376	18	6·807	34	0·8532
3	3·903	22	5·674	35	0·4160
4	4·986	26	4·241	$35\frac{1}{2}$	0·2043
6	6·380	30	2·610	$35\frac{3}{4}$	0·1012
10	7·526	32	1·736	36	0·0004

The correction to k_1 is everywhere subordinate. In the last step from $35\frac{3}{4}$ to 36 no correction to k_1 is applied.

There is a little difficulty in tracing by this method and with full accuracy the final progress to zero when $\xi' = 36$. If any doubt be left, it may be removed by applying the former method to the course from 35 to 36, using the value of f appropriate to 35 and the terminal tangent as the representative of the curve for F. From this it appears that even if U' at $\xi' = 35$ were as great as $0{\cdot}7468$, the moving particle could not pass $\xi' = 36$. The conclusion is that even in this extreme case of $s = 6$ the solution exists, and that a wave of permanent regime is possible. Further, from (99) we see that since U and μ' are both positive, $d\xi/dx$ is positive throughout, and the transition from the one density to the other takes place *without alternation.*

After what has been proved little doubt could remain but that a solution is possible when s has any value lower than 6. I have, however, thought it desirable to add a rough calculation (rough on account of the relative magnitude of the steps) for the case of $s = 3$:—

ξ'	1	2	3	5	7	8	$8\frac{1}{2}$	9
U'	0·00	0·90	1·19	1·26	0·82	0·40	0·19	0·01

It is a question of some importance to consider what is the thickness of the transitional layer in the waves of uniform regime which have been proved to be possible. Mathematically speaking, the transition occupies an infinite space; but if we understand the expression to refer to a transition approximately complete, the thickness involved is finite, and indeed extremely small. Reference to (98) shows that x is of the order μ', or μ/m, or $\mu/\rho u$, where u is the velocity of the wave. For the present purpose we

may take u as equal to the usual velocity of sound, *i.e.* 3×10^4 cm. per second. For air under ordinary conditions the value of μ/ρ in C.G.S. measure is 0·13; so that x is of the order $\frac{1}{3} \times 10^{-5}$ cm. That the transitional layer is in fact extremely thin is proved by such photographs as those of Boys, of the aerial wave of approximate discontinuity which advances in front of a modern rifle bullet; but that according to calculation this thickness should be well below the microscopic limit may well occasion surprise.

Resistance to Motion through Air at High Velocities.

According to the adiabatic law the pressures and velocities in a compressible fluid free from external force, see (47), are related by

$$\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} = 1 + \frac{\gamma-1}{2} \frac{u_1^2 - u_2^2}{a_1^2}, \tag{134}$$

in which p_1, ρ_1, u_1 denote the pressure, density, and velocity at one point of the path; p_2, ρ_2, u_2 the corresponding quantities at another point. Also $a_1^2 = \gamma p_1/\rho_1$, so that a_1 is the velocity of infinitesimal disturbances in the condition (1). In an early paper* I suggested the application of this formula to bodies moving through air at high velocities. Regarding the obstacle as stationary and the fluid in motion with velocity u_1 and pressure p_1, the pressure p_2 corresponding to the loss of this velocity is given by putting $u_2 = 0$ in (133), a_1 being the ordinary velocity of sound. This is the pressure which should obtain at the axial point on the nose of a symmetrical bullet, and although this value in strictness represents the *maximum* pressure, the analogy of an incompressible fluid suggests that the mean pressure on a flat surface would not be greatly inferior. But in a recent discussion,† Mr. Mallock has shown that this formula immensely overestimates the resistance actually experienced by a bullet, and (so far as I am aware) the discrepancy remains unexplained.

If indeed the adiabatic law really prevailed throughout, there could be no escape from the conclusion formulated. A consideration of the photographs by Boys‡ will suggest the required explanation. At a short constant distance in front of the bullet there is an aerial bore, or place of approximate discontinuity, Along the axis, the fluid moving up to the bullet changes its density, and therefore pressure and temperature, *suddenly*, so that there is here a special opportunity for viscosity and heat-conduction to take effect. The pressures and velocities on the two sides of the bore are

* 'Phil. Mag.,' 1876, vol. 2, p. 430; 'Scientific Papers,' vol. 1, p. 289.

† 'Roy. Soc. Proc.,' A, 1907, vol. 79, p. 266.

‡ 'Nature,' 1893, vol. 47, p. 440. The particular photograph reproduced by Mallock does not exhibit well the feature in question.

related, not according to the adiabatic law, but according to Rankine's law already discussed. The changes which occur may be separated into two stages. The first is the sudden one in which the fluid passes from the atmospheric condition p_0, ρ_0, with velocity u_0 to the condition denoted by p_1, ρ_1, u_1. After passing the bore the fluid changes gradually according to the adiabatic law already stated until at the nose of the bullet the condition is represented by p_2, ρ_2, with $u_2 = 0$.

We are now in a position to calculate the final pressure p_2. For the first stage we have Rankine's formula (67), making

$$\rho_0 u_0^2 = \tfrac{1}{2}(\gamma-1)p_0+\tfrac{1}{2}(\gamma+1)p_1,$$

or, if $a^2 = \gamma p_0/\rho_0$,

$$\frac{p_1}{p_0} = \frac{2\gamma}{\gamma+1}\frac{u_0^2}{a^2}-\frac{\gamma-1}{\gamma+1}, \tag{135}$$

determining the pressure just inside the bore in terms of u_0 (the velocity of the bullet through quiescent air) and a, the ordinary velocity of sound. When $u_0 = a$, $p_1 = p_0$. For values of u_0 less than a, the first stage does not exist and we may suppose $p_1 = p_0$, $u_1 = u_0$.

In the second stage we use (134) with $u_2 = 0$. Thus

$$\left(\frac{p_2}{p_1}\right)^{\frac{\gamma-1}{\gamma}} = 1+\frac{\gamma-1}{2}\frac{\rho_1 u_1^2}{\gamma p_1}, \tag{136}$$

in which, by a formula analogous to (66),

$$\rho_1 u_1^2 = \tfrac{1}{2}(\gamma+1)p_0+\tfrac{1}{2}(\gamma-1)p_1.$$

Hence

$$\left(\frac{p_2}{p_0}\right)^{\frac{\gamma-1}{\gamma}} = \frac{(\gamma+1)^2}{4\gamma}\left(\frac{p_1}{p_0}\right)^{\frac{\gamma-1}{\gamma}}\left\{1+\frac{\gamma-1}{\gamma+1}\frac{p_0}{p_1}\right\}. \tag{137}$$

When $u_0 > a$, p_1/p_0 is to be calculated from (135). When the resulting value is substituted in (137), p_2/p_0 is determined.

When $u_0 < a$, we have simply

$$\left(\frac{p_2}{p_0}\right)^{\frac{\gamma-1}{\gamma}} = 1+\frac{\gamma-1}{2}\frac{u_0^2}{a^2}. \tag{138}$$

If u_0/a be *small*, (138) reduces to

$$\frac{p_2}{p_0} = 1+\frac{\rho_0 u_0^2}{2p_0},$$

or

$$p_2-p_0 = \tfrac{1}{2}\rho_0 u_0^2, \tag{139}$$

as for an incompressible fluid.

When $u_0 = a$, both systems give

$$\left(\frac{p_2}{p_0}\right)^{\frac{\gamma-1}{\gamma}} = \frac{\gamma+1}{2}. \tag{140}$$

When u_0/a is *large*, the second terms on the right of (135) and (137) may be neglected, and we obtain

$$\frac{p_2}{p_0} = \frac{\gamma+1}{2}\frac{u_0^2}{a^2}\left\{\frac{(\gamma+1)^2}{4\gamma}\right\}^{\frac{1}{\gamma-1}}; \tag{141}$$

or, when we put $\gamma = 1{\cdot}41$,

$$p_2/p_0 = 1{\cdot}30\,u_0^2/a^2. \tag{142}$$

The following are some corresponding values of p_2/p_0 and u_0/a, calculated from (135), (137), with $\gamma = 1{\cdot}41$:—

u_0/a	1	2	3	4
p_2/p_0......	1·90	4·49	11·7	20·7

From this point onwards the approximate formula (142) may suffice. The values found are in good agreement with Mr. Mallock's curve.

The question as to the linear interval between the bore and the nose of the bullet cannot be answered from the results of the present paper. In strictly one-dimensional motion, the bore and the plane wall constituting the obstacle could not move at the same speed.